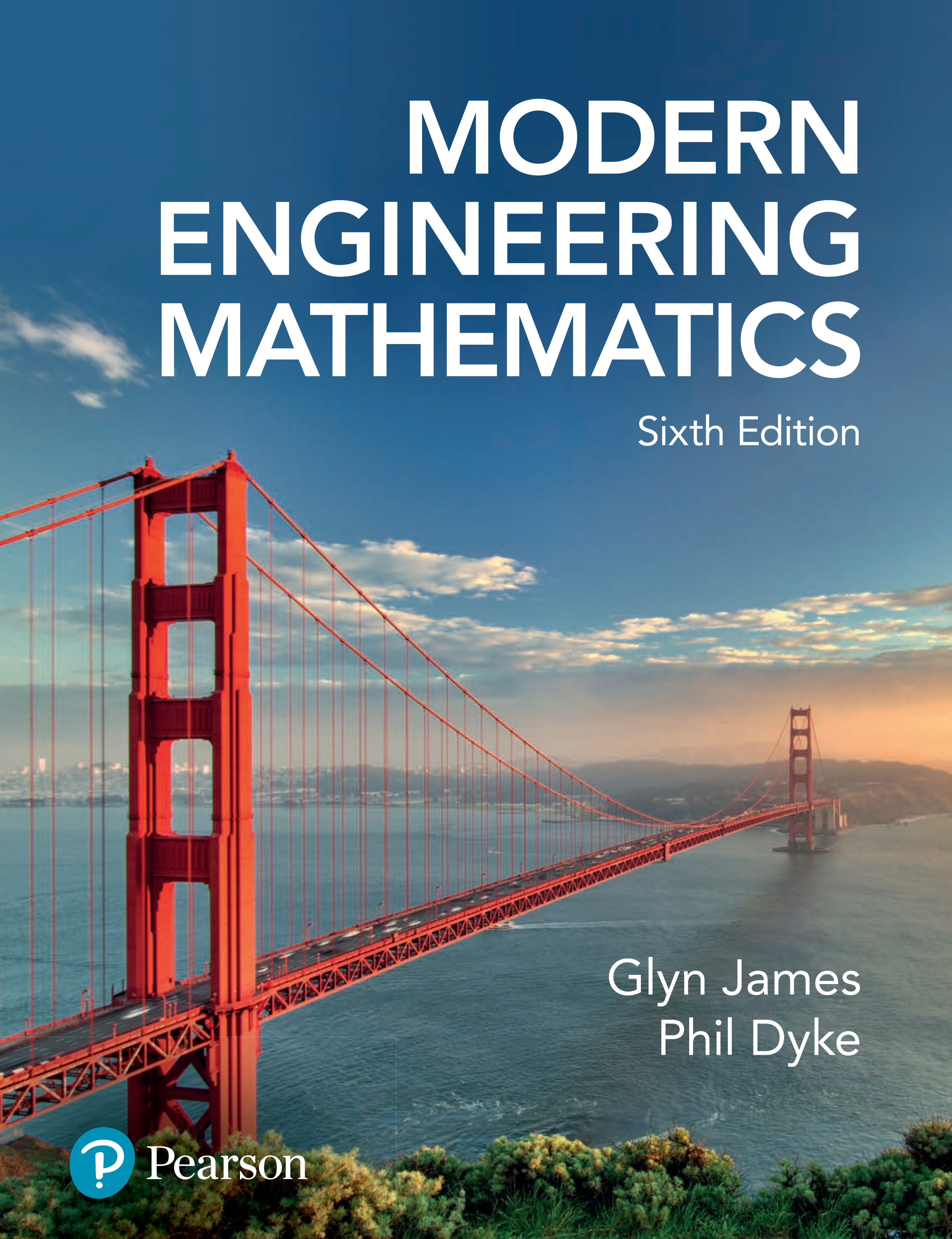

MODERN ENGINEERING MATHEMATICS

Sixth Edition

Glyn James
Phil Dyke

Pearson

Modern Engineering Mathematics

Modern Engineering Mathematics

Sixth Edition

Glyn James *Coventry University*
Phil Dyke *University of Plymouth*
and
John Searl *University of Edinburgh*
Matthew Craven *University of Plymouth*
Yinghui Wei *University of Plymouth*

Pearson

Harlow, England • London • New York • Boston • San Francisco • Toronto • Sydney • Dubai • Singapore • Hong Kong
Tokyo • Seoul • Taipei • New Delhi • Cape Town • São Paulo • Mexico City • Madrid • Amsterdam • Munich • Paris • Milan

PEARSON EDUCATION LIMITED
KAO Two
KAO Park
Harlow CM17 9SR
United Kingdom
Tel: +44 (0)1279 623623
Web: www.pearson.com/uk

Previously published 1992, 1996, 2001, 2008 (print)
Fourth edition with MyMathLab published 2010 (print)
Fifth edition published 2015 (print and electronic)
Sixth edition published 2020 (print and electronic)

ISBN: 978-1-292-25349-7 (print)
978-1-292-25353-4 (PDF)
978-1-292-25355-8 (ePub)

British Library Cataloguing-in-Publication Data
A catalogue record for the print edition is available from the British Library

Library of Congress Cataloging-in-Publication Data
Names: James, Glyn, author.
Title: Modern engineering mathematics/Glyn James, Coventry University, Phil Dyke University of Plymouth, and John Searl, University of Edinburgh, Matthew Craven, University of Plymouth, Yinghui Wei, University of Plymouth.
Description: Sixth edition. | Harlow, England; Hoboken, NJ : Pearson, 2020.
Identifiers: LCCN 2019052464 (print) | LCCN 2019052465 (ebook) | ISBN 9781292253497 (paperback) | ISBN 9781292253534 (ebook)
Subjects: LCSH: Engineering mathematics.
Classification: LCC TA330 .J36 2020 (print) | LCC TA330 (ebook) | DDC 510.2/462--dc23
LC record available at https://lccn.loc.gov/2019052464
LC ebook record available at https://lccn.loc.gov/2019052465

10 9 8 7 6 5 4 3 2 1
24 23 22 21 20

Cover credit: Prab S/500px Prime/Getty Images
Print edition typeset in 10/12pt Times LT Pro by Spi Global
Printed in Slovakia by Neografia

NOTE THAT ANY PAGE CROSS REFERENCES REFER TO THE PRINT EDITION

Contents

Companion Website

For open-access **student resources** specifically written to complement this textbook and support your learning, please visit **go.pearson.com/uk/he/resources**

ON THE WEBSITE

Lecturer Resources

For password-protected online resources tailored to support the use of this textbook in teaching, please visit **go.pearson.com/uk/he/resources**

Preface

The first edition of this book appeared in 1992; this is the sixth edition and there have been a few changes, mostly a few corrections and additions, but also more substantive changes to Chapter 13 Data Handling and Probability Theory. Echoing the words of my predecessor Professor Glyn James, the range of material covered in this sixth edition is regarded as appropriate for a first-level core studies course in mathematics for undergraduate courses in all engineering disciplines. Whilst designed primarily for use by engineering students it is believed that the book is also highly suitable for students of the physical sciences and applied mathematics. Additional material appropriate for second-level undergraduate core studies, or possibly elective studies for some engineering disciplines, is contained in the companion text *Advanced Modern Engineering Mathematics*.

The objective of the authoring team remains that of achieving a balance between the development of understanding and the mastering of solution techniques, with the emphasis being on the development of the student's ability to use mathematics with understanding to solve engineering problems. Consequently, the book is not a collection of recipes and techniques designed to teach students to solve routine exercises, nor is mathematical rigour introduced for its own sake. To achieve the desired objective the text contains:

- Worked examples
  Approximately 500 worked examples, many of which incorporate mathematical models and are designed both to provide relevance and to reinforce the role of mathematics in various branches of engineering. In response to feedback from users, additional worked examples have been incorporated within this revised edition.
- Applications
  To provide further exposure to the use of mathematical models in engineering practice, each chapter contains sections on engineering applications. These sections form an ideal framework for individual, or group, case study assignments leading to a written report and/or oral presentation, thereby helping to develop the skills of mathematical modelling necessary to prepare for the more open-ended modelling exercises at a later stage of the course.
- Exercises
  There are numerous exercise sections throughout the text, and at the end of each chapter there is a comprehensive set of review exercises. While many of the exercise problems are designed to develop skills in mathematical techniques,

others are designed to develop understanding and to encourage learning by doing, and some are of an open-ended nature. This book contains over 1200 exercises and answers to all the questions are given. It is hoped that this provision, together with the large number of worked examples and style of presentation, also make the book suitable for private or directed study. Again in response to feedback from users, the frequency of exercise sections has been increased and additional questions have been added to many of the sections.

- Numerical methods
  Recognizing the increasing use of numerical methods in engineering practice, which often complement the use of analytical methods in analysis and design and are of ultimate relevance when solving complex engineering problems, there is wide agreement that they should be integrated within the mathematics curriculum. Consequently the treatment of numerical methods is integrated within the analytical work throughout the book.

The position of software use is an important aspect of engineering education. The decision has been taken to use mainly MATLAB but also, in later chapters, MAPLE. Students are encouraged to make intelligent use of software, and where appropriate codes are included, but there is a health warning. The pace of technology shows little signs of lessening, and so in the space of six years, the likely time lapse before a new edition of this text, it is probable that software will continue to be updated, probably annually. There is therefore a real risk that much coding, though correct and working at the time of publication, could be broken by these updates. Therefore, in this edition the decision has been made not to overemphasize specific code but to direct students to the Companion Website or to general principles instead. The software packages, particularly MAPLE, have become easier to use without the need for programming skills. Much is menu driven these days. Here is more from Glyn on the subject that is still true:

> Students are strongly encouraged to use one of these packages to check the answers to the examples and exercises. It is stressed that the MATLAB (and a few MAPLE) inserts are not intended to be a first introduction of the package to students; it is anticipated that they will receive an introductory course elsewhere and will be made aware of the excellent ‘help’ facility available. The purpose of incorporating the inserts is not only to improve efficiency in the use of the package but also to provide a facility to help develop a better understanding of the related mathematics. Whilst use of such packages takes the tedium out of arithmetic and algebraic manipulations it is important that they are used to enhance understanding and not to avoid it. It is recognized that not all users of the text will have access to either MATLAB or MAPLE, and consequently all the inserts are highlighted and can be ‘omitted’ without loss of continuity in developing the subject content.

Throughout the text two icons are used:

- An open screen indicates that use of a software package would be useful (for example, for checking solutions) but not essential.
- A closed screen indicates that the use of a software package is essential or highly desirable.

Specific changes in this sixth edition are an improvement in many of the diagrams, taking advantage of present-day software, and modernization of the examples and language. Also, Chapter 13 Data Handling and Probability Theory has been significantly modernized by interfacing the presentation with the very powerful software package R. It is free; simply search for 'R Software' and download it. I have been much aided in getting this edition ready for publication by my hardworking colleagues Matthew, John and Yinghui who now comprise the team.

Feedback from users of the previous edition on the subject content has been favourable, and consequently no new chapters have been introduced. However, in response to the feedback, chapters have been reviewed and amended/updated accordingly. Whilst subject content at this level has not changed much over the years the mode of delivery is being driven by developments in computer technology. Consequently there has been a shift towards online teaching and learning, coupled with student self-study programmes. In support of such programmes, worked examples and exercise sections are seen by many as the backbone of the text. Consequently in this new edition emphasis is given to strengthening the 'Worked Examples' throughout the text and increasing the frequency and number of questions in the 'Exercise Sections'. This has involved the restructuring, sometimes significantly, of material within individual chapters.

A comprehensive Solutions Manual is obtainable free of charge to lecturers using this textbook. It will be available for download online at go.pearson.com/uk/he/resources.

Also available online is a set of 'Refresher Units' covering topics students should have encountered at school but may not have used for some time.

This text is also paired with a MyLab™ - a teaching and learning platform that empowers you to reach every student. By combining trusted author content with digital tools and a flexible platform, MyLab personalizes the learning experience and improves results for each student. MyLab Math for this textbook has over 1150 questions to assign to your students, including exercises requiring different types of mathematics applications for a variety of industry types. Note that students require a course ID and an access card in order to use MyLab Math (see inside front cover for more information or contact your Pearson account manager at the link go.pearson.com/findarep).

Acknowledgements

The authoring team is extremely grateful to all the reviewers and users of the text who have provided valuable comments on previous editions of this book. Most of this has been highly constructive and very much appreciated. The team has continued to enjoy the full support of a very enthusiastic production team at Pearson Education and wishes to thank all those concerned.

Phil Dyke
Plymouth
and
Glyn James
Coventry
July 2019

About the authors

New authors Matthew Craven and Yinghui Wei join one of the original authors John Searl under the new editor, also one of the original authors, Phil Dyke, to produce this the sixth edition of Modern Engineering Mathematics.

Phil Dyke is Professor of Applied Mathematics at the University of Plymouth. He was a Head of School for twenty-two years, eighteen of these as Head of Mathematics and Statistics. He has over forty-five years' teaching and research experience in Higher Education, much of this teaching engineering students not only mathematics but also marine and coastal engineering. Apart from his contributions to both *Modern Engineering Mathematics* and *Advanced Modern Engineering Mathematics* he is the author of eleven other textbooks ranging in topic from advanced calculus, Laplace transforms and Fourier series to mechanics and marine physics. He is now semi-retired, but still teaches, is involved in research, and writes. He is a Fellow of the Institute of Mathematics and its Applications.

Matthew Craven is a Lecturer in Applied Mathematics at the University of Plymouth. For fifteen years, he has taught foundation year, postgraduate and everything in between. He is also part of the author team for the 5th edition of the companion text, *Advanced Modern Engineering Mathematics*. He has research interests in computational simulation, real-world operational research, high performance computing and optimization.

Yinghui Wei is an Associate Professor of Statistics at the University of Plymouth. She has taught probability and statistics modules for mathematics programmes as well as for programmes in other subject areas, including engineering, business and medicine. She has broad research interests in statistical modelling, data analysis and evidence synthesis.

John Searl was Director of the Edinburgh Centre for Mathematical Education at the University of Edinburgh before his retirement. As well as lecturing on mathematical education, he taught service courses for engineers and scientists. His most recent research concerned the development of learning environments that make for the effective learning of mathematics for 16–20 year olds. As an applied mathematician he worked collaboratively with (amongst others) engineers, physicists, biologists and pharmacologists, he is keen to develop problem-solving skills of students and to provide them with opportunities to display their mathematical knowledge within a variety of practical contexts. The contexts develop the extended reasoning needed in all fields of engineering.

The original editor was **Glyn James** who retired as Dean of the School of Mathematical and Information Sciences at Coventry University in 2001 and then became Emeritus Professor in Mathematics at the University. He graduated from the University College of Wales, Cardiff in the late 1950s, obtaining first-class honours degrees in both Mathematics and Chemistry. He obtained a PhD in Engineering Science in 1971 as an external student of the University of Warwick. He was employed at Coventry in 1964 and held the position of the Head of Mathematics Department prior to his appointment as Dean in 1992. His research interests were in control theory and its applications to industrial problems. He also had a keen interest in mathematical education, particularly in relation to the teaching of engineering mathematics and mathematical modelling. He was co-chairman of the European Mathematics Working Group established by the European Society for Engineering Education (SEFI) in 1982, a past chairman of the Education Committee of the Institute of Mathematics and its Applications (IMA), and a member of the Royal Society Mathematics Education Subcommittee. In 1995 he was chairman of the Working Group that produced the report *Mathematics Matters in Engineering* on behalf of the professional bodies in engineering and mathematics within the UK. He was also a member of the editorial/advisory board of three international journals. He published numerous papers and was co-editor of five books on various aspects of mathematical modelling. He was a past Vice-President of the IMA and also served a period as Honorary Secretary of the Institute. He was a Chartered Mathematician and a Fellow of the IMA. Sadly, Glyn James passed away in October 2019 during the production of this edition; his enthusiastic input was sorely missed, but this and its companion text remain a fitting legacy.

The original authors are David Burley, Dick Clements, Jerry Wright together with Phil Dyke and John Searl. The short biographies that are not here can be found in the previous editions.

1 Number, Algebra and Geometry

Chapter 1	Contents	
1.1	Introduction	2
1.2	Number and arithmetic	2
1.3	Algebra	14
1.4	Geometry	36
1.5	Number and accuracy	47
1.6	Engineering applications	57
1.7	Review exercises (1–25)	59

1.1 Introduction

Mathematics plays an important role in our lives. It is used in everyday activities from buying food to organizing maintenance schedules for aircraft. Through applications developed in various cultural and historical contexts, mathematics has been one of the decisive factors in shaping the modern world. It continues to grow and to find new uses, particularly in engineering and technology, from electronic circuit design to machine learning.

Mathematics provides a powerful, concise and unambiguous way of organizing and communicating information. It is a means by which aspects of the physical universe can be explained and predicted. It is a problem-solving activity supported by a body of knowledge. Mathematics consists of facts, concepts, skills and thinking processes – aspects that are closely interrelated. It is a hierarchical subject in that new ideas and skills are developed from existing ones. This sometimes makes it a difficult subject for learners who, at every stage of their mathematical development, need to have ready recall of material learned earlier.

In the first two chapters we shall summarize the concepts and techniques that most students will already understand and we shall extend them into further developments in mathematics. There are four key areas of which students will already have considerable knowledge.

- numbers
- algebra
- geometry
- functions

These areas are vital to making progress in engineering mathematics (indeed, they will solve many important problems in engineering). Here we will aim to consolidate that knowledge, to make it more precise and to develop it. In this first chapter we will deal with the first three topics; functions are considered next (see Chapter 2).

1.2 Number and arithmetic

1.2.1 Number line

Mathematics has grown from primitive arithmetic and geometry into a vast body of knowledge. The most ancient mathematical skill is counting, using, in the first instance, the natural numbers and later the integers. The term **natural numbers** commonly refers to the set $\mathbb{N} = \{1, 2, 3, \ldots\}$, and the term **integers** to the set $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$. The integers can be represented as equally spaced points on a line called the **number line** as shown in Figure 1.1. In a computer the integers can be stored exactly. The set of all points (not just those representing integers) on the number line represents the **real numbers** (so named to distinguish them from the complex numbers, which are

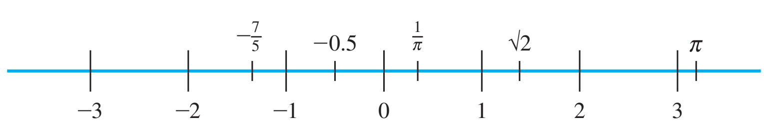

Figure 1.1 The number line.

discussed in Chapter 3). The set of real numbers is denoted by $\mathbb{R}$. The general real number is usually denoted by the letter x and we write 'x in $\mathbb{R}$', meaning x is a real number. A real number that can be written as the ratio of two integers, like $\frac{3}{2}$ or $-\frac{7}{5}$, is called a **rational number**. Other numbers, like $\sqrt{2}$ and π, that cannot be expressed in that way are called **irrational numbers**. In a computer the real numbers can be stored only to a limited number of figures. This is a basic difference between the ways in which computers treat integers and real numbers, and is the reason why the computer languages commonly used by engineers distinguish between integer values and variables on the one hand and real number values and variables on the other.

1.2.2 Representation of numbers

For everyday purposes we use a system of representation based on ten **numerals**: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. These ten symbols are sufficient to represent all numbers if a **position notation** is adopted. For whole numbers this means that, starting from the right-hand end of the number, the least significant end, the figures represent the number of units, tens, hundreds, thousands, and so on. Thus one thousand, three hundred and sixty-five is represented by 1365, and two hundred and nine is represented by 209. Notice the role of the 0 in the latter example, acting as a position keeper. The use of a decimal point makes it possible to represent fractions as well as whole numbers. This system uses ten symbols. The number system is said to be 'to base ten' and is called the **decimal** system. Other bases are possible: for example, the Babylonians used a number system to base sixty, a fact that still influences our measurement of time. In some societies a number system evolved with more than one base, a survival of which can be seen in imperial measures (inches, feet, yards, …). For some applications it is more convenient to use a base other than ten. Early electronic computers used **binary** numbers (to base two); modern computers use **hexadecimal** numbers (to base sixteen). For elementary (pen-and-paper) arithmetic a representation to base twelve would be more convenient than the usual decimal notation because twelve has more integer divisors (2, 3, 4, 6) than ten (2, 5).

In a decimal number the positions to the left of the decimal point represent units (10^0), tens (10^1), hundreds (10^2) and so on, while those to the right of the decimal point represent tenths (10^{-1}), hundredths (10^{-2}) and so on. Thus, for example,

$$\begin{array}{cccccc} 2 & 1 & 4 & \cdot & 3 & 6 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ 10^2 & 10^1 & 10^0 & & 10^{-1} & 10^{-2} \end{array}$$

so

$$\begin{aligned} 214.36 &= 2(10^2) + 1(10^1) + 4(10^0) + 3(\tfrac{1}{10}) + 6(\tfrac{1}{100}) \\ &= 200 + 10 + 4 + \tfrac{3}{10} + \tfrac{6}{100} \\ &= \tfrac{21436}{100} = \tfrac{5359}{25} \end{aligned}$$

In other number bases the pattern is the same: in base b the position values are b^0, b^1, b^2, … and b^{-1}, b^{-2}, … . Thus in binary (base two) the position values are units, twos, fours, eights, sixteens and so on, and halves, quarters, eighths and so on. In hexadecimal (base sixteen) the position values are units, sixteens, two hundred and fifty-sixes and so on, and sixteenths, two hundred and fifty-sixths and so on.

Example 1.1

Write (a) the binary number 1011101_2 as a decimal number and (b) the decimal number 115_{10} as a binary number.

Solution

(a) $1011101_2 = 1(2^6) + 0(2^5) + 1(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$

$= 64_{10} + 0 + 16_{10} + 8_{10} + 4_{10} + 0 + 1_{10}$

$= 93_{10}$

(b) We achieve the conversion to binary by repeated division by 2. Thus

$115 \div 2 = 57$	remainder 1	(2^0)
$57 \div 2 = 28$	remainder 1	(2^1)
$28 \div 2 = 14$	remainder 0	(2^2)
$14 \div 2 = 7$	remainder 0	(2^3)
$7 \div 2 = 3$	remainder 1	(2^4)
$3 \div 2 = 1$	remainder 1	(2^5)
$1 \div 2 = 0$	remainder 1	(2^6)

so that

$115_{10} = 1110011_2$

Example 1.2

Represent the numbers (a) two hundred and one, (b) two hundred and seventy-five, (c) five and three-quarters and (d) one-third in

(i) decimal form using the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;

(ii) binary form using the figures 0, 1;

(iii) duodecimal (base twelve) form using the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, Δ, ε.

Solution

(a) two hundred and one

(i) = 2 (hundreds) + 0 (tens) and 1 (units) = 201_{10}

(ii) = 1 (one hundred and twenty-eight) + 1 (sixty-four) + 1 (eight) + 1 (unit) = 11001001_2

(iii) =1 (gross) + 4 (dozens) + 9 (units) = 149_{12}

Here the subscripts 10, 2, 12 indicate the number base.

(b) two hundred and seventy-five

(i) = 2 (hundreds) + 7 (tens) + 5 (units) = 275_{10}

(ii) = 1 (two hundred and fifty-six) + 1 (sixteen) + 1 (two) + 1 (unit) = 100010011_2

(iii) = 1 (gross) + 10 (dozens) + eleven (units) = $1\Delta\varepsilon_{12}$
(Δ represents ten and ε represents eleven)

(c) five and three-quarters

(i) = 5 (units) + 7 (tenths) + 5 (hundredths) = 5.75_{10}

(ii) = 1 (four) + 1 (unit) + 1 (half) + 1 (quarter) = 101.11_2

(iii) = 5 (units) + 9 (twelfths) = 5.9_{12}

(d) one-third

(i) = 3 (tenths) + 3 (hundredths) + 3 (thousandths) + … = $0.333\ldots_{10}$

(ii) = 1 (quarter) + 1 (sixteenth) + 1 (sixty-fourth) + … = $0.010101\ldots_2$

(iii) = 4 (twelfths) = 0.4_{12}

1.2.3 Rules of arithmetic

The basic arithmetical operations of addition, subtraction, multiplication and division are performed subject to the **Fundamental Rules of Arithmetic**. For any three numbers a, b and c:

(a1) the commutative law of addition

$$a + b = b + a$$

(a2) the commutative law of multiplication

$$a \times b = b \times a$$

(b1) the associative law of addition

$$(a + b) + c = a + (b + c)$$

(b2) the associative law of multiplication

$$(a \times b) \times c = a \times (b \times c)$$

(c1) the distributive law of multiplication over addition and subtraction

$$(a + b) \times c = (a \times c) + (b \times c)$$

$$(a - b) \times c = (a \times c) - (b \times c)$$

(c2) the distributive law of division over addition and subtraction

$$(a + b) \div c = (a \div c) + (b \div c)$$

$$(a - b) \div c = (a \div c) - (b \div c)$$

Here the brackets indicate which operation is performed first. These operations are called **binary** operations because they associate with every two members of the set of real numbers a unique third member; for example,

$$2 + 5 = 7 \quad \text{and} \quad 3 \times 6 = 18$$

Example 1.3 Find the value of $(100 + 20 + 3) \times 456$.

Solution Using the distributive law we have

$$\begin{aligned}(100 + 20 + 3) \times 456 &= 100 \times 456 + 20 \times 456 + 3 \times 456 \\ &= 45\,600 + 9120 + 1368 = 56\,088\end{aligned}$$

Here 100×456 has been evaluated as

$$100 \times 400 + 100 \times 50 + 100 \times 6$$

and similarly 20×456 and 3×456.

This, of course, is normally set out in the traditional school arithmetic way:

```
   456
   123 ×
 ------
 1 368
 9 120
45 600
------
56 088
```

Example 1.4 Rewrite $(a + b) \times (c + d)$ as the sum of products.

Solution Using the distributive law we have

$$\begin{aligned}(a + b) \times (c + d) &= a \times (c + d) + b \times (c + d) \\ &= (c + d) \times a + (c + d) \times b \\ &= c \times a + d \times a + c \times b + d \times b \\ &= a \times c + a \times d + b \times c + b \times d\end{aligned}$$

applying the commutative laws several times.

A further operation used with real numbers is that of **powering**. For example, $a \times a$ is written as a^2, and $a \times a \times a$ is written as a^3. In general the product of n a's where n is a positive integer is written as a^n. (Here the n is called the **index** or **exponent**.) Operations with powering also obey simple rules:

$$a^n \times a^m = a^{n+m} \qquad \textbf{(1.1a)}$$

$$a^n \div a^m = a^{n-m} \qquad \textbf{(1.1b)}$$

$$(a^n)^m = a^{nm} \qquad \textbf{(1.1c)}$$

From rule (1.1b) it follows, by setting $n = m$ and $a \neq 0$, that $a^0 = 1$. It is also convention to take $0^0 = 1$. The process of powering can be extended to include the fractional powers like $a^{1/2}$. Using rule (1.1c),

$$(a^{1/n})^n = a^{n/n} = a^1$$

and we see that

$$a^{1/n} = {}^n\sqrt{a}$$

the nth root of a. Also, we can define a^{-m} using rule (1.1b) with $n = 0$, giving

$$1 \div a^m = a^{-m}, \qquad a \neq 0$$

Thus a^{-m} is the reciprocal of a^m. In contrast with the binary operations $+$, $\times$, $-$ and $\div$, which operate on two numbers, the powering operation $(\ )^r$ operates on just one element and is consequently called a **unary** operation. Notice that the fractional power

$$a^{m/n} = ({}^n\sqrt{a})^m = {}^n\sqrt{(a^m)}$$

is the nth root of a^m. If n is an even integer, then $a^{m/n}$ is not defined when a is negative. When ${}^n\sqrt{a}$ is an irrational number then such a root is called a **surd**.

Numbers like $\sqrt{2}$ were described by the Greeks as **a-logos**, without a ratio number. An Arabic translator took the alternative meaning 'without a word' and used the Arabic word for 'deaf', which subsequently became **surdus**, Latin for deaf, when translated from Arabic to Latin in the mid-twelfth century.

Example 1.5

Find the values of

(a) $27^{1/3}$ (b) $(-8)^{2/3}$ (c) $16^{-3/2}$

(d) $(-2)^{-2}$ (e) $(-1/8)^{-2/3}$ (f) $(9)^{-1/2}$

Solution

(a) $27^{1/3} = {}^3\sqrt{27} = 3$

(b) $(-8)^{2/3} = ({}^3\sqrt{(-8)})^2 = (-2)^2 = 4$

(c) $16^{-3/2} = (16^{1/2})^{-3} = (4)^{-3} = \frac{1}{4^3} = \frac{1}{64}$

(d) $(-2)^{-2} = \dfrac{1}{(-2)^2} = \frac{1}{4}$

(e) $(-1/8)^{-2/3} = [{}^3\sqrt{(-1/8)}]^{-2} = [{}^3\sqrt{(-1)}/{}^3\sqrt{(8)}]^{-2} = [-1/2]^{-2} = 4$

(f) $(9)^{-1/2} = (3)^{-1} = \frac{1}{3}$

Example 1.6

Express (a) in terms of $\sqrt{2}$ and simplify (b) to (f).

(a) $\sqrt{18} + \sqrt{32} - \sqrt{50}$ (b) $6/\sqrt{2}$ (c) $(1 - \sqrt{3})(1 + \sqrt{3})$

(d) $\dfrac{2}{1 - \sqrt{3}}$ (e) $(1 + \sqrt{6})(1 - \sqrt{6})$ (f) $\dfrac{1 - \sqrt{2}}{1 + \sqrt{6}}$

Solution (a) $\sqrt{18} = \sqrt{(2 \times 9)} = \sqrt{2} \times \sqrt{9} = 3\sqrt{2}$

$\sqrt{32} = \sqrt{(2 \times 16)} = \sqrt{2} \times \sqrt{16} = 4\sqrt{2}$

$\sqrt{50} = \sqrt{(2 \times 25)} = \sqrt{2} \times \sqrt{25} = 5\sqrt{2}$

Thus $\sqrt{18} + \sqrt{32} - \sqrt{50} = 2\sqrt{2}$.

(b) $6/\sqrt{2} = 3 \times 2/\sqrt{2}$

Since $2 = \sqrt{2} \times \sqrt{2}$, we have $6/\sqrt{2} = 3\sqrt{2}$.

(c) $(1 - \sqrt{3})(1 + \sqrt{3}) = 1 + \sqrt{3} - \sqrt{3} - 3 = -2$

(d) Using the result of part (c), $\dfrac{2}{1-\sqrt{3}}$ can be simplified by multiplying 'top and bottom' by $1 + \sqrt{3}$ (notice the sign change in front of the $\sqrt{}$). Thus

$$\frac{2}{1-\sqrt{3}} = \frac{2(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$= \frac{2(1+\sqrt{3})}{1-3}$$

$$= -1 - \sqrt{3}$$

(e) $(1 + \sqrt{6})(1 - \sqrt{6}) = 1 - \sqrt{6} + \sqrt{6} - 6 = -5$

(f) Using the same technique as in part (d) we have

$$\frac{1-\sqrt{2}}{1+\sqrt{6}} = \frac{(1-\sqrt{2})(1-\sqrt{6})}{(1+\sqrt{6})(1-\sqrt{6})}$$

$$= \frac{1-\sqrt{2}-\sqrt{6}+\sqrt{12}}{1-6}$$

$$= -(1 - \sqrt{2} - \sqrt{6} + 2\sqrt{3})/5$$

This process of expressing the irrational number so that all of the surds are in the numerator is called **rationalization**.

When evaluating arithmetical expressions the following rules of precedence are observed:

- the powering operation $(\ )^r$ is performed first
- then multiplication $\times$ and/or division $\div$
- then addition $+$ and/or subtraction $-$

When two operators of equal precedence are adjacent in an expression the left-hand operation is performed first. For example,

$$12 - 4 + 13 = 8 + 13 = 21$$

and

$$15 \div 3 \times 2 = 5 \times 2 = 10$$

The precedence rules are overridden by brackets; thus

$$12 - (4 + 13) = 12 - 17 = -5$$

and

$$15 \div (3 \times 2) = 15 \div 6 = 2.5$$

This order of precedence is commonly referred to as BODMAS/BIDMAS (meaning: brackets, order/index, multiplication, addition, subtraction).

Example 1.7 Evaluate $7 - 5 \times 3 \div 2^2$.

Solution Following the rules of precedence, we have

$$7 - 5 \times 3 \div 2^2 = 7 - 5 \times 3 \div 4 = 7 - 15 \div 4 = 7 - 3.75 = 3.25$$

1.2.4 Exercises

1. Find the decimal equivalent of 110110.101_2.

2. Find the binary and octal (base eight) equivalents of the decimal number 16 321. Obtain a simple rule that relates these two representations of the number, and hence write down the octal equivalent of 1011100101101_2.

3. Find the binary and octal equivalents of the decimal number 30.6. Does the rule obtained in Question 2 still apply?

4. Use binary arithmetic to evaluate

   (a) $100011.011_2 + 1011.001_2$

   (b) $111.10011_2 \times 10.111_2$

5. Simplify the following expressions, giving the answers with positive indices and without brackets:

   (a) $2^3 \times 2^{-4}$ (b) $2^3 \div 2^{-4}$ (c) $(2^3)^{-4}$

   (d) $3^{1/3} \times 3^{5/3}$ (e) $(36)^{-1/2}$ (f) $16^{3/4}$

6. The expression $7 - 2 \times 3^2 + 8$ may be evaluated using the usual implicit rules of precedence. It could be rewritten as $((7 - (2 \times (3^2))) + 8)$ using brackets to make the precedence explicit. Similarly rewrite the following expressions in fully bracketed form:

   (a) $21 + 4 \times 3 \div 2$

   (b) $17 - 6^{2+3}$

   (c) $4 \times 2^3 - 7 \div 6 \times 2$

   (d) $2 \times 3 - 6 \div 4 + 3^{2-5}$

7. Express the following in the form $x + y\sqrt{2}$ with x and y rational numbers:

   (a) $(7 + 5\sqrt{2})^3$ (b) $(2 + \sqrt{2})^4$

   (c) $\sqrt[3]{(7 + 5\sqrt{2})}$ (d) $\sqrt{(\frac{11}{2} - 3\sqrt{2})}$

8. Show that

$$\frac{1}{a + b\sqrt{c}} = \frac{a - b\sqrt{c}}{a^2 - b^2c}$$

   Hence express the following numbers in the form $x + y\sqrt{n}$ where x and y are rational numbers and n is an integer:

   (a) $\dfrac{1}{7 + 5\sqrt{2}}$ (b) $\dfrac{2 + 3\sqrt{2}}{9 - 7\sqrt{2}}$

   (c) $\dfrac{4 - 2\sqrt{3}}{7 - 3\sqrt{3}}$ (d) $\dfrac{2 + 4\sqrt{5}}{4 - \sqrt{5}}$

9. Find the difference between 2 and the squares of

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$$

   (a) Verify that successive terms of the sequence stand in relation to each other as m/n does to $(m + 2n)/(m + n)$.

   (b) Verify that if m/n is a good approximation to $\sqrt{2}$ then $(m + 2n)/(m + n)$ is a better one, and that the errors in the two cases are in opposite directions.

   (c) Find the next three terms of the above sequence.